Recent Works on Optimization for Signal Processing

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Outline

- Basic Concepts
- Filter Designs
- Image Resizing
- Conclusions
- References
- Questions and Answers

Basic Concepts

- What are Optimization Problems?
  - Finding the value of $\mathbf{x}$ such that the functional value at $\mathbf{x}$ is either minimum or maximum.
  - Minimization problem

$$\min_{\mathbf{x}\in\Re^d} f(\mathbf{x})$$

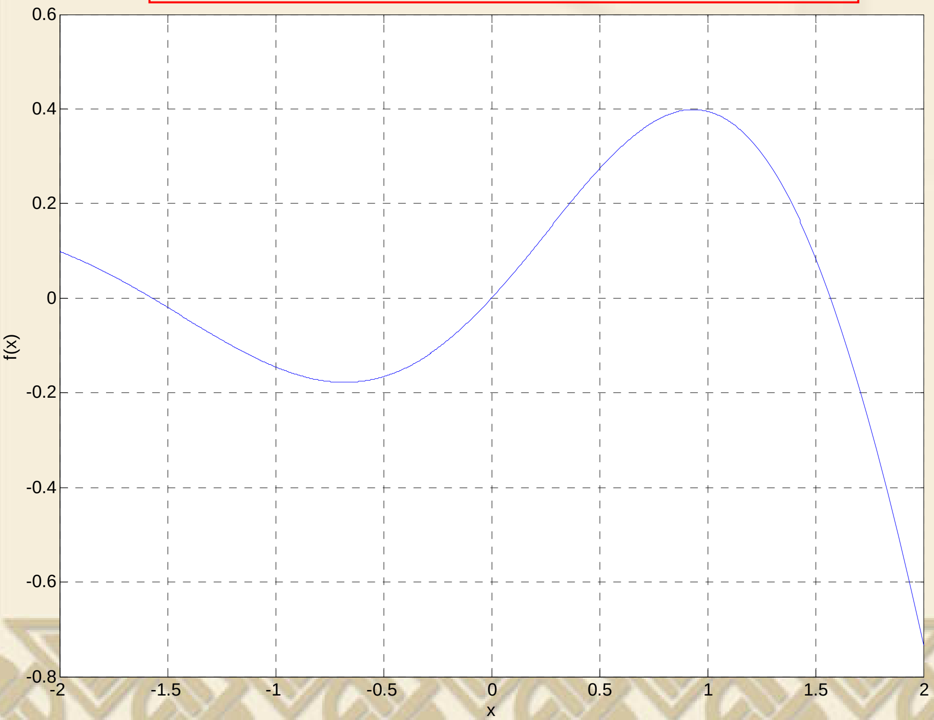
  - Maximization problem

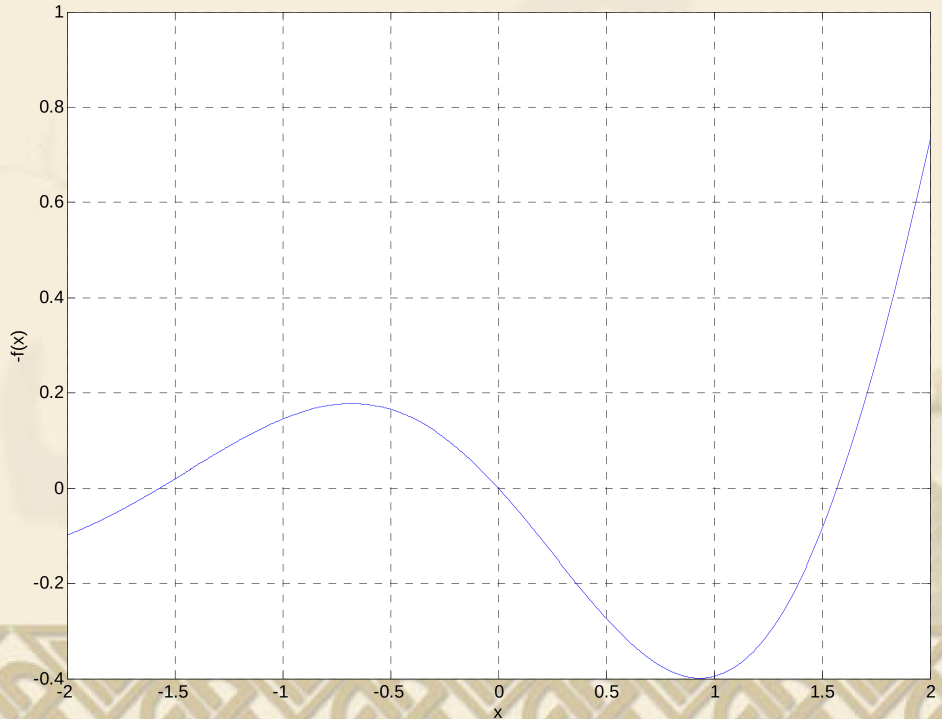
$$\max_{\mathbf{x}\in\Re^d} f(\mathbf{x})$$

Basic Concepts

- Relationship Between Maximization Problems and Minimization Problems

$$\max_{\mathbf{x}\in\Re^d} f(\mathbf{x}) \Leftrightarrow \min_{\mathbf{x}\in\Re^d} - f(\mathbf{x})$$





Basic Concepts

- Constrained and Unconstrained Optimization Problems
  - Unconstrained optimization problem

$$\min_{\mathbf{x}\in\Re^d} f(\mathbf{x})$$

  - Constrained optimization problem

$$\min_{\mathbf{x}\in\Re^d} f(\mathbf{x})$$

subject to $g_i(\mathbf{x}) \le 0$ for $i = 1,2,\cdots,M$ (inequality constraints)

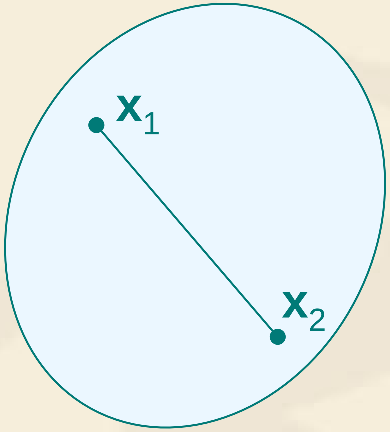
$h_i(\mathbf{x}) = 0$ for $i = 1,2,\cdots,N$ (equality constraints)

$p_i(\mathbf{x},\omega) \le 0$ for $i = 1,2,\cdots,K$ and $\forall \omega \in \Omega$ (functional inequality constraints)

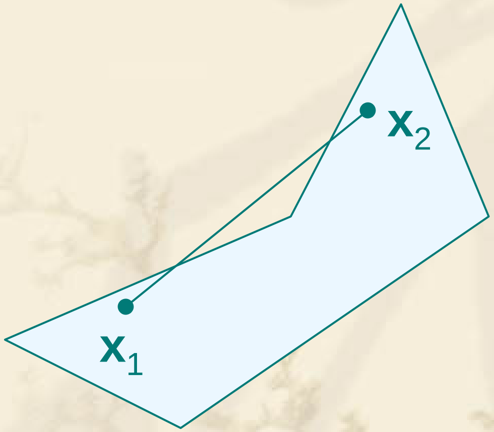
Basic Concepts

- Convex and Nonconvex Optimization Problems
  - Convex sets

$\forall \mathbf{x}_1, \mathbf{x}_2 \in S$ and $\forall \lambda \in [0,1]$, $\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in S$



(a) Convex set



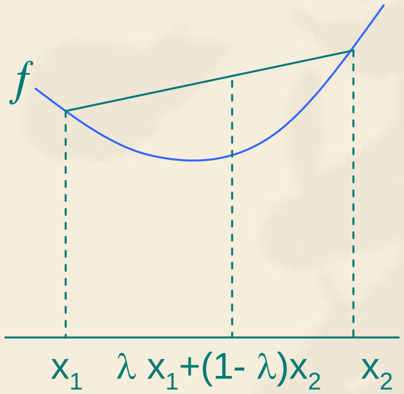
(b) Nonconvex set

Basic Concepts

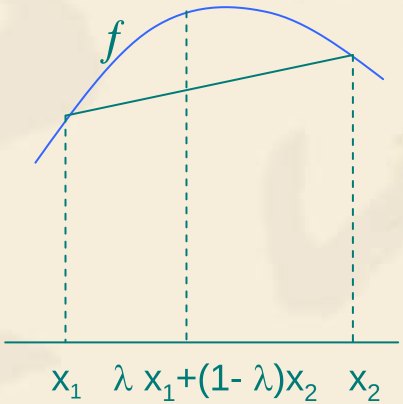
- Convex and Nonconvex Optimization Problems
  - Convex functions

Let $f : S \to \Re$, where S is a nonempty convex set. The function f is said to be convex on S if

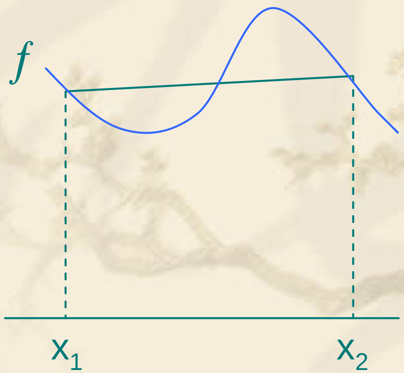
$\forall \mathbf{x}_1, \mathbf{x}_2 \in S$ and $\forall \lambda \in [0,1]$, $f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \le \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)$



convex function



concave function



neither convex nor concave

Basic Concepts

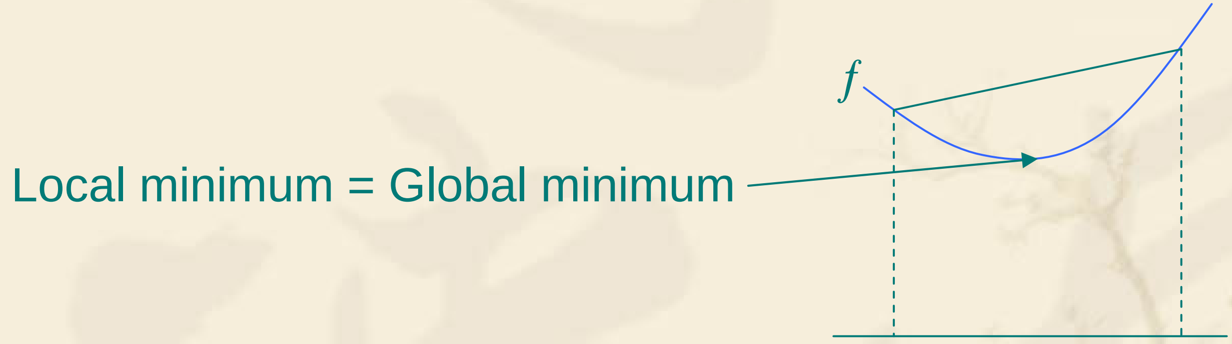
- Convex and Nonconvex Optimization Problems
  - Feasible set

$$\Psi \equiv \left\{ \begin{array}{l} \mathbf{x} : g_i(\mathbf{x}) \le 0 \text{ for } i = 1,2,\cdots,M, \\ \quad h_i(\mathbf{x}) = 0 \text{ for } i = 1,2,\cdots,N, \\ \quad p_i(\mathbf{x},\omega) \le 0 \text{ for } i = 1,2,\cdots,K \text{ and } \forall \omega \in \Omega \end{array} \right\}$$

  - Convex optimization problem
    - Feasible set is convex and f is convex.
  - Nonconvex optimization problem
    - Feasible set is not convex, or f is not convex, or neither.

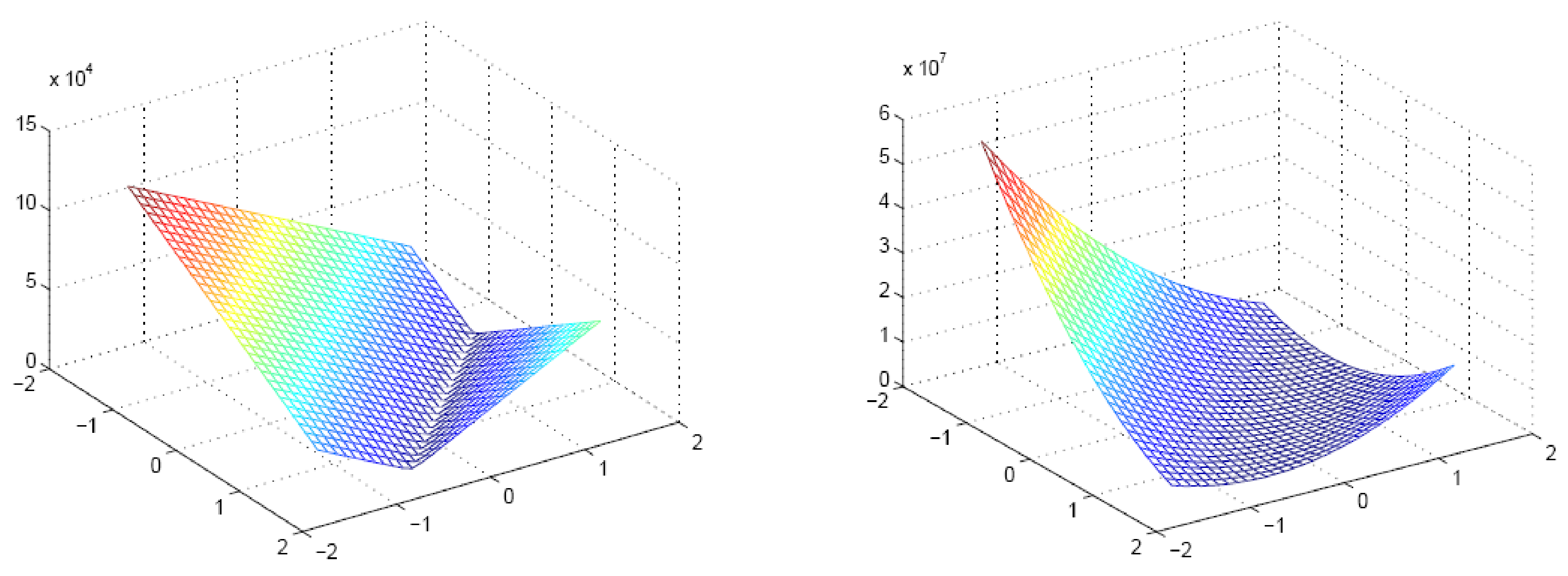
Basic Concepts

- Convex and Nonconvex Optimization Problems
  - If the optimization problem is convex, then any local minimum is a global minimum.



Basic Concepts

- Smooth and Nonsmooth Optimization Problems
  - Smooth optimization problems
    - f is differentiable.
  - Nonsmooth optimization problems
    - f is not differentiable.



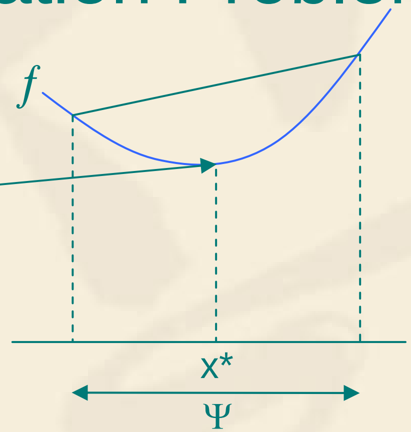
Basic Concepts

- Smooth and Nonsmooth Optimization Problems
  - For smooth optimization problems, if $\mathbf{x}^*$ is a local minimum of f and $\mathbf{x}^* \in \Psi$, then $\mathbf{x}^*$ is a stationary point. If $\mathbf{x}^*$ is a stationary point, $\mathbf{x}^* \in \Psi$ and the Hessian matrix evaluated at $\mathbf{x}^*$ is positive definite, then $\mathbf{x}^*$ is a local minimum.

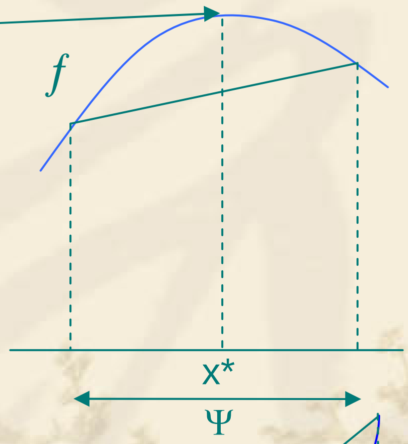
Basic Concepts

- Smooth and Nonsmooth Optimization Problems

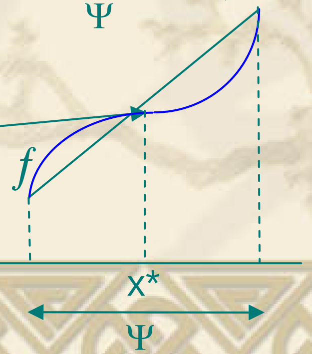
Local minimum $\Rightarrow$ stationary point
A stationary point and convex $\Rightarrow$ local minimum



Local maximum $\Rightarrow$ stationary point
A stationary point and concave $\Rightarrow$ local maximum

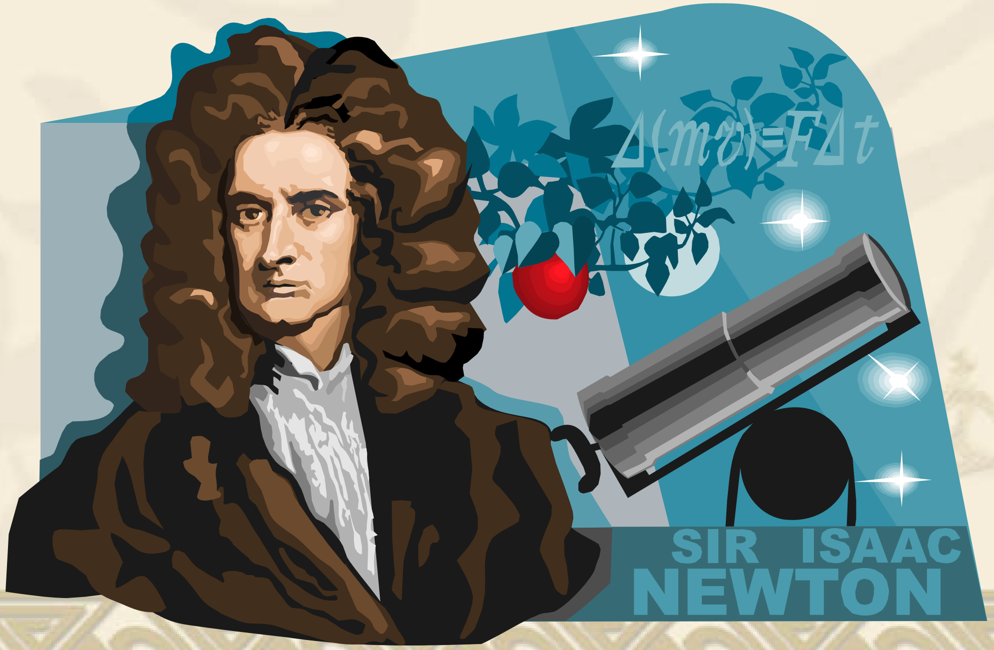


Point of reflection $\Rightarrow$ stationary point
A stationary point which is twice differentiable, but neither convex nor concave $\Rightarrow$ point of reflection



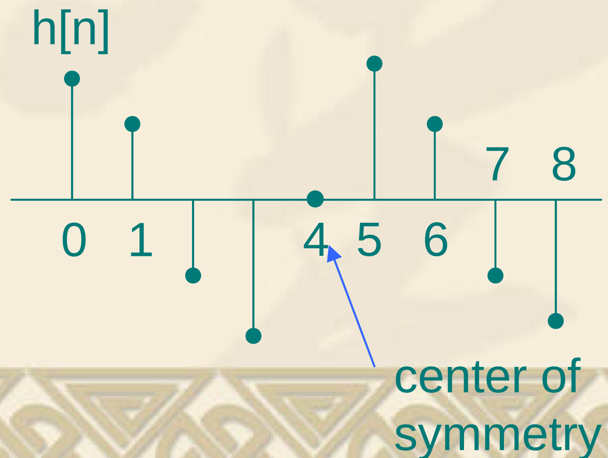
Basic Concepts

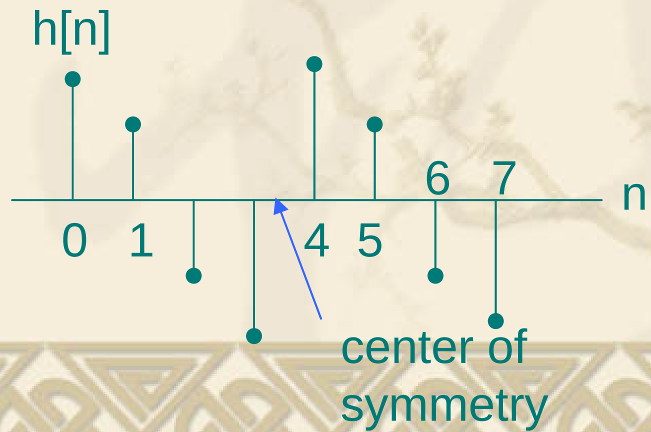
- Smooth and Nonsmooth Optimization Problems
  - Local optimal solution of smooth problems could be found by Newton's method and steepest decent method.



Filter Designs

- Finite Impulse Response (FIR) Linear Phase Anti-symmetric Filter Design Problems
  - For N is odd, $\begin{cases} h(k) = -h(N-1-k), & k = 0,1,2,\cdots,\frac{N-3}{2} \\ h\left(\frac{N-1}{2}\right) = 0 \end{cases}$
  - For N is even, $h(k) = -h(N-1-k)$ for $k = 0,1,2,\cdots,\frac{N}{2}-1$





Filter Designs

- FIR Linear Phase Anti-symmetric Filter Design Problems
  - Denote

$$\mathbf{x} \equiv \begin{cases} \left[a_1, \quad a_2, \quad \cdots, \quad a_{\frac{N-1}{2}} \right]^T, & N \text{ is odd} \\ \left[a_1, \quad a_2, \quad \cdots, \quad a_{\frac{N}{2}} \right]^T, & N \text{ is even} \end{cases}$$

where

$$a_n \equiv \begin{cases} 2h\left(\dfrac{N-1}{2} - n\right), & N \text{ is odd and } n = 1, 2, \cdots, \dfrac{N-1}{2} \\ 2h\left(\dfrac{N}{2} - n\right), & N \text{ is even and } n = 1, 2, \cdots, \dfrac{N}{2} \end{cases}$$

Filter Designs

- FIR Linear Phase Anti-symmetric Filter Design Problems
  - Denote

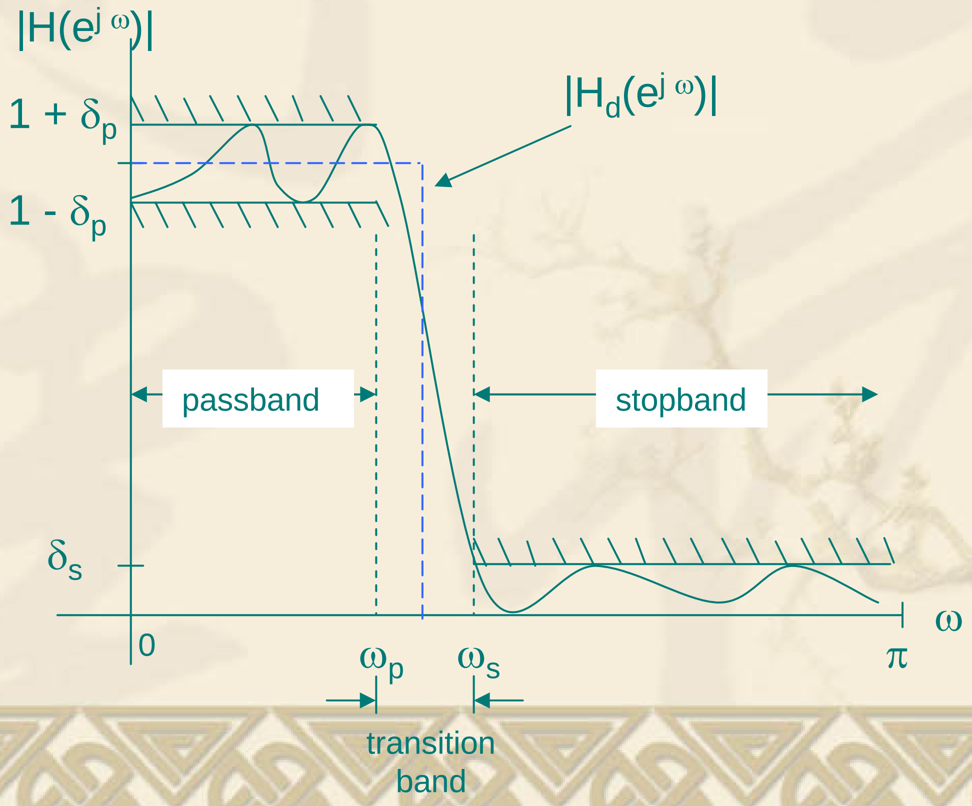
$$\boldsymbol{\eta}(\omega) \equiv \begin{cases} \left[\sin\omega, \quad \sin 2\omega, \quad \cdots, \quad \sin\left(\left(\frac{N-1}{2}\right)\omega\right) \right]^T, & N \text{ is odd} \\ \left[\sin\frac{\omega}{2}, \quad \sin\frac{3\omega}{2}, \quad \cdots, \quad \sin\left(\left(\frac{N-1}{2}\right)\omega\right) \right]^T, & N \text{ is even} \end{cases}$$

$$H_0(\omega) \equiv (\boldsymbol{\eta}(\omega))^T \mathbf{x}$$

  - Then $H(\omega) = \sum_{k=0}^{N-1} h(n) e^{-jn\omega} = je^{-j\omega\left(\frac{N-1}{2}\right)} H_0(\omega)$

Filter Designs

- FIR Linear Phase Anti-symmetric Filter Design Problems
  - Objective: Minimize the weighted total ripple energy subject to the weighted peak constraint.



Filter Designs

- FIR Linear Phase Anti-symmetric Filter Design Problems

$$J(\mathbf{x}) \equiv \int_{B_d} W(\omega)\left|H_0(\omega) - D(\omega)\right|^2 d\omega = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\,\mathbf{x} + \mathbf{b}^T\mathbf{x} + p$$

where

$$\mathbf{Q} = 2\int_{B_d} W(\omega)\boldsymbol{\eta}(\omega)\left(\boldsymbol{\eta}(\omega)\right)^T d\omega$$

$$\mathbf{b} = -2\int_{B_d} W(\omega) D(\omega)\boldsymbol{\eta}(\omega) d\omega$$

$$p = \int_{B_d} W(\omega)\left(D(\omega)\right)^2 d\omega$$

$$W(\omega) > 0 \;\; \forall \omega \in B_d$$

Filter Designs

- FIR Linear Phase Anti-symmetric Filter Design Problems

$$W(\omega)\left| H_0(\omega) - D(\omega) \right| \le \delta \quad \forall \omega \in B_d$$

$$\Leftrightarrow \quad \mathbf{A}(\omega)\,\mathbf{x} \le \mathbf{c}(\omega) \quad \forall \omega \in B_d$$

where $\mathbf{A}(\omega) = W(\omega)\left[\boldsymbol{\eta}(\omega), \quad -\boldsymbol{\eta}(\omega)\right]^T \qquad \forall \omega \in B_d$

$\mathbf{c}(\omega) = \left[D(\omega)W(\omega) + \delta, \quad \delta - D(\omega)W(\omega)\right]^T \quad \forall \omega \in B_d$

Problem (**P**) $\quad \min\limits_{\mathbf{x}} \quad J(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\,\mathbf{x} + \mathbf{b}^T\mathbf{x} + p$

Subject to $\quad \mathbf{g}(\mathbf{x}, \omega) = \mathbf{A}(\omega)\,\mathbf{x} - \mathbf{c}(\omega) \le \mathbf{0} \quad \forall \omega \in B_d$

Filter Designs

- Challenges of Functional Inequality Constrained Optimization Problems
  - The domain of functional inequalities is $\Re^d \times \Omega$.
  - $\Rightarrow$infinite number of constraints.
  - How to guarantee that these infinite number of constraints are satisfied?
  - How to solve these problems efficiently?

Filter Designs

- Solutions for Solving Functional Inequality Constrained Optimization Problems
  - Dual parameterization approach
    - For smooth and convex optimization problems, by discretizing the index set Ω with finite number of elements, denoted as ω_i for $i=1,2,\cdots,k$, and introducing parameters λ_i for $i=1,2,\cdots,k$, then the following two optimization problems are equivalent:

$$\min_{\mathbf{x}\in\Re^d} \quad f(\mathbf{x})$$

subject to $\mathbf{g}(\mathbf{x},\omega)\le\mathbf{0} \quad \forall\omega\in\Omega$

$$\min_{\mathbf{x}} \max_{(\boldsymbol{\omega},\boldsymbol{\lambda})\in\Re^{k\times k}} f(\mathbf{x})+\sum_{i=1}^{k}\boldsymbol{\lambda}_i^T\mathbf{g}(\mathbf{x},\omega_i)$$

subject to $\boldsymbol{\lambda}_i\ge 0$ for $i=1,2,\cdots,k$

$\omega_i\in\Omega$ for $i=1,2,\cdots,k$

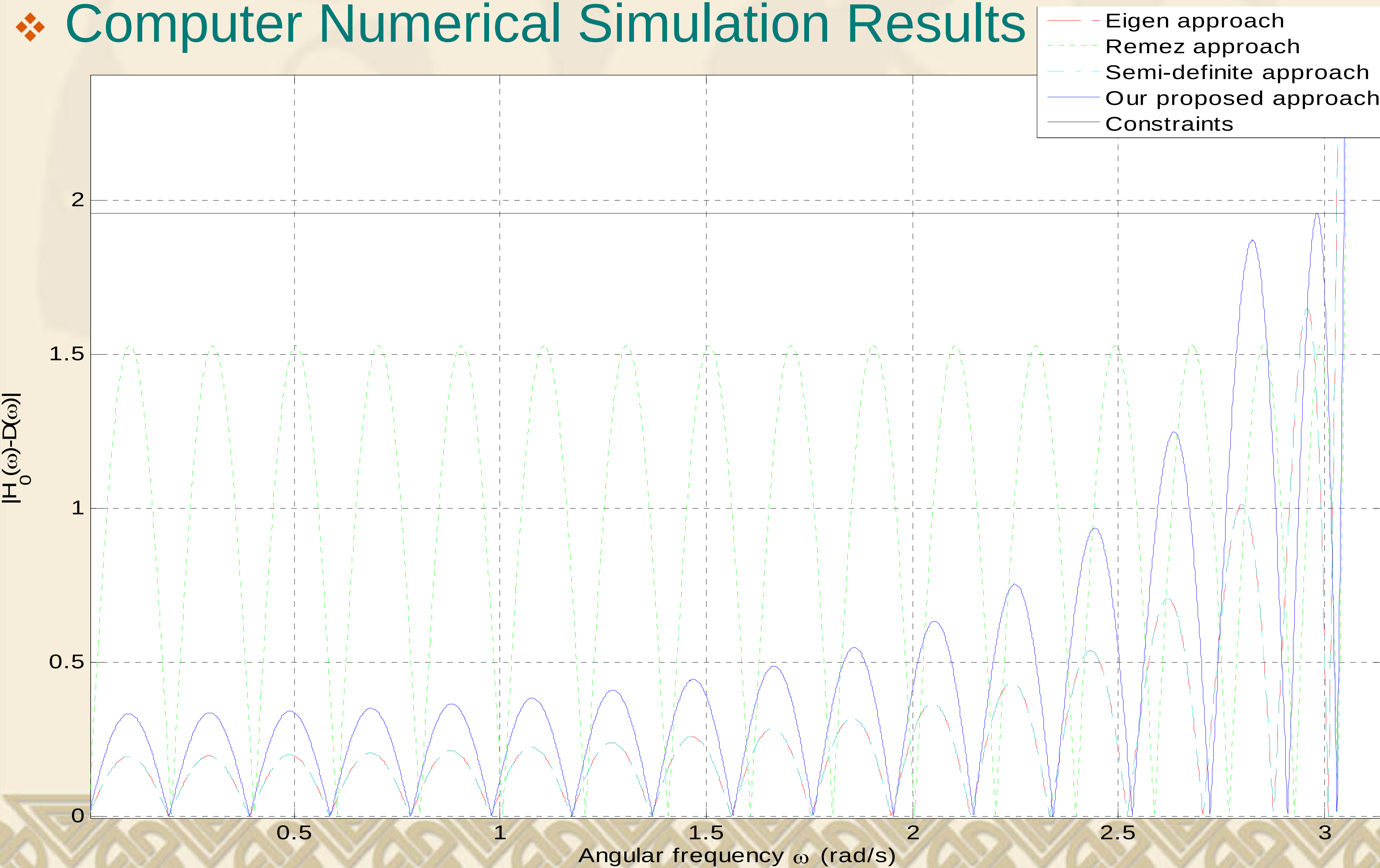
$$\nabla_{\mathbf{x}} f(\mathbf{x})+\sum_{i=1}^{k}\boldsymbol{\lambda}_i^T\nabla_{\mathbf{x}}\mathbf{g}(\mathbf{x},\omega_i)=\mathbf{0}$$

Filter Designs

- Solutions for Solving Functional Inequality Constrained Optimization Problems
  - Dual parameterization approach
    - guarantees that the obtained global minimum satisfies the required functional inequality constraint if the feasible set is non-empty.
    - The maximum number of required discretization points is less than or equal to $d+2$. Hence, this optimization problem can be solved efficiently.

Filter Designs

- Computer Numerical Simulation Results



Filter Designs

- Infinite Impulse Response (IIR) Filter Design Problems
  - Objective: Minimize the weighted total ripple energy subject to the weighted peak constraint.

$$H(\omega)=\frac{e^{-jD\omega}\sum_{m=0}^{M}b_m e^{-jm\omega}}{1+\sum_{n=1}^{N}a_n e^{-jn\omega}}$$

$$\left|\frac{e^{-jD\omega}\sum_{m=0}^{M}b_m e^{-jm\omega}}{1+\sum_{n=1}^{N}a_n e^{-jn\omega}}\right|^2 \approx \left(\tilde{H}(\omega)\right)^2$$

$$E(\omega)\equiv\left|e^{-jD\omega}\sum_{m=0}^{M}b_m e^{-jm\omega}\right|^2-\left(\tilde{H}(\omega)\right)^2\left|1+\sum_{n=1}^{N}a_n e^{-jn\omega}\right|^2$$

Filter Designs

- IIR Filter Design Problems

$$\mathbf{x}_n \equiv \begin{bmatrix} b_0, & b_1, & \cdots, & b_M \end{bmatrix}^T$$

$$\mathbf{x}_d \equiv \begin{bmatrix} a_1, & a_2, & \cdots, & a_N \end{bmatrix}^T$$

$$\boldsymbol{\eta}_n(\omega) \equiv \begin{bmatrix} 1, & e^{-j\omega}, & \cdots, & e^{-jM\omega} \end{bmatrix}^T$$

$$\boldsymbol{\eta}_d(\omega) \equiv \begin{bmatrix} e^{-j\omega}, & e^{-j2\omega}, & \cdots, & e^{-jN\omega} \end{bmatrix}^T$$

$$E(\omega) = \left| \left(\boldsymbol{\eta}_n(\omega)\right)^T \mathbf{x}_n \right|^2 - \left(\tilde{H}(\omega)\right)^2 \left| 1 + \left(\boldsymbol{\eta}_d(\omega)\right)^T \mathbf{x}_d \right|^2$$

$$\tilde{J}(\mathbf{x}_n, \mathbf{x}_d) \equiv \int_{B_P \cup B_S} W(\omega) \left| E(\omega) \right| d\omega$$

where $W(\omega) > 0 \quad \forall \omega \in B_P \cup B_S$

Filter Designs

- IIR Filter Design Problems

$$\operatorname{Re}\left(1+\left(\boldsymbol{\eta}_d(\omega)\right)^T \mathbf{x}_d\right)>0 \quad \forall \omega \in[-\pi, \pi]$$

$$\tilde{W}(\omega)|E(\omega)| \leq \tilde{\delta}(\omega) \quad \forall \omega \in B_P \cup B_S$$

where $\tilde{W}(\omega)>0 \quad \forall \omega \in B_P \cup B_S$

$$\tilde{W}(\omega)|E(\omega)| \leq \tilde{\delta}(\omega) \forall \omega \in B_P \cup B_S \Leftrightarrow \begin{array}{l} \tilde{W}(\omega) E(\omega) \leq \tilde{\delta}(\omega) \forall \omega \in B_P \cup B_S \\ \text{and} -\tilde{\delta}(\omega) \leq \tilde{W}(\omega) E(\omega) \forall \omega \in B_P \cup B_S \end{array}$$

Problem (**Q**) $\min_{(\mathbf{x}_n, \mathbf{x}_d)} \tilde{J}(\mathbf{x}_n, \mathbf{x}_d) \equiv \int_{B_P \cup B_S} W(\omega)|E(\omega)| d\omega$

subject to

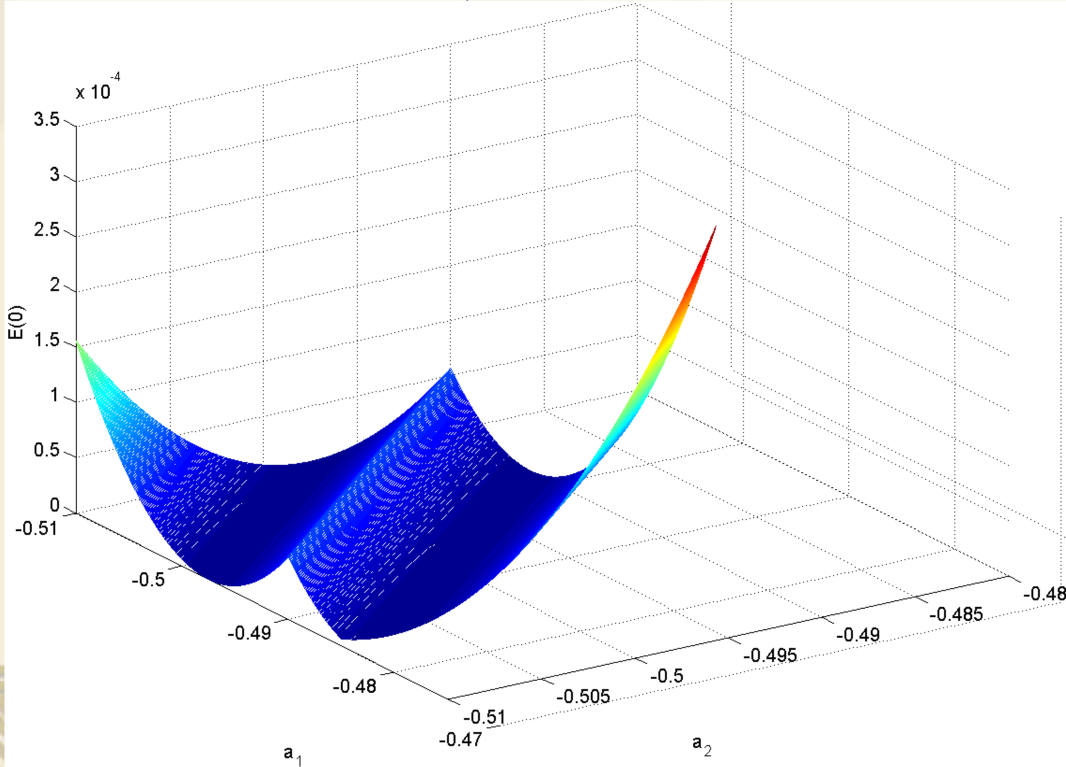
$$\begin{aligned} &\tilde{g}_1(\mathbf{x}_n, \mathbf{x}_d, \omega) \equiv \tilde{W}(\omega) E(\omega)-\tilde{\delta}(\omega) \leq 0 && \forall \omega \in B_P \cup B_S \\ &\tilde{g}_2(\mathbf{x}_n, \mathbf{x}_d, \omega) \equiv -\tilde{W}(\omega) E(\omega)-\tilde{\delta}(\omega) \leq 0 && \forall \omega \in B_P \cup B_S \\ &\tilde{g}_3(\mathbf{x}_d, \omega) \equiv \operatorname{Re}\left(1+\left(\boldsymbol{\eta}_d(\omega)\right)^T \mathbf{x}_d\right)>0 && \forall \omega \in[-\pi, \pi] \end{aligned}$$

Filter Designs

- Challenges of Nonsmooth Functional Inequality Constrained Optimization Problems
  - Consider the following IIR filter design problem with the error function $$E(\omega) \equiv \left(\left| \sum_{m=0}^{2} b_m e^{-jm\omega} \right| - \left| 1 + \sum_{n=1}^{2} a_n e^{-jn\omega} \right| \right)^2$$

    where

$$b_0 = 2.816335701763035 \times 10^{-3}$$
$$b_1 = 1.877557134508662 \times 10^{-3}$$
$$b_2 = 2.816335701763063 \times 10^{-3}$$



Filter Designs

- Challenges of Nonsmooth Functional Inequality Constrained Optimization Problems
  - The optimization problem is a nonsmooth functional inequality constrained optimization problems, in which Newton’s method and steepest decent method cannot be applied for solving the problem.

Filter Designs

- Solutions for Solving Nonsmooth Optimization Problems
  - Since $\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}=\begin{cases}0 & \tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)\le 0\\ \text{positive value} & \tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)>0\end{cases}$
  - By defining

$$\hat{g}_1(\mathbf{x}_n,\mathbf{x}_d)\equiv\int_{B_P\cup B_S}\left(\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\right)^2 d\omega$$

  - We have

$$\hat{g}_1(\mathbf{x}_n,\mathbf{x}_d)=\begin{cases}0 & \forall\omega\in B_P\cup B_S,\ \tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)\le 0\\ \text{positive value} & \exists\omega\in B_P\cup B_S,\ \tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)>0\end{cases}$$

$$\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)\le 0\ \forall\omega\in B_P\cup B_S \Leftrightarrow \hat{g}_1(\mathbf{x}_n,\mathbf{x}_d)=0$$

Filter Designs

- Solutions for Solving Nonsmooth Optimization Problems

$$\left(\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\right)^2=\begin{cases}0 & \tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)\le 0\\ \left(\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)\right)^2 & \tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)>0\end{cases}$$

$$\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\left(\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\right)^2=\begin{cases}\mathbf{0} & \tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)<0\\ 2\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega) & \tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)>0\end{cases}$$

  - As $2\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)=\mathbf{0}$ when $\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)=0$. $\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\left(\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\right)^2$ is continuous at $\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)=0$.
  - As $2\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)=\mathbf{0}$ when $\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)<0$ and $2\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)=2\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)$ when $\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)>0$

    We have, $\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\left(\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\right)^2=2\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)$

Filter Designs

- Solutions for Solving Nonsmooth Optimization Problems
  - Consequently, we have

$$\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\hat{g}_1(\mathbf{x}_n,\mathbf{x}_d)=2\int_{B_P\cup B_S}\max\{\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\tilde{g}_1(\mathbf{x}_n,\mathbf{x}_d,\omega)d\omega$$

  - Similarly, define $\hat{g}_2(\mathbf{x}_n,\mathbf{x}_d)\equiv\int_{B_P\cup B_S}\left(\max\{\tilde{g}_2(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\right)^2 d\omega$

$$\hat{g}_3(\mathbf{x}_d)\equiv\int_{[-\pi,\pi]}\left(\max\{\tilde{g}_3(\mathbf{x}_d,\omega),0\}\right)^2 d\omega$$

  - We have

$$\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\hat{g}_2(\mathbf{x}_n,\mathbf{x}_d)=2\int_{B_P\cup B_S}\max\{\tilde{g}_2(\mathbf{x}_n,\mathbf{x}_d,\omega),0\}\nabla_{(\mathbf{x}_n,\mathbf{x}_d)}\tilde{g}_2(\mathbf{x}_n,\mathbf{x}_d,\omega)d\omega$$

$$\nabla_{\mathbf{x}_d}\hat{g}_3(\mathbf{x}_d)=2\int_{[-\pi,\pi]}\max\{\tilde{g}_3(\mathbf{x}_d,\omega),0\}\nabla_{\mathbf{x}_d}\tilde{g}_3(\mathbf{x}_d,\omega)d\omega$$

Filter Designs

- Solutions for Solving Nonsmooth Optimization Problems
  - Now the problem become the following equality constrained optimization problem.

$$\min_{(\mathbf{x}_n,\mathbf{x}_d)} \quad \tilde{J}(\mathbf{x}_n,\mathbf{x}_d) \equiv \int_{B_P \cup B_S} W(\omega)\left|E(\omega)\right| d\omega$$

subject to

$$\hat{g}_1(\mathbf{x}_n,\mathbf{x}_d)=0$$

$$\hat{g}_2(\mathbf{x}_n,\mathbf{x}_d)=0$$

$$\hat{g}_3(\mathbf{x}_d)=0$$

$\forall \omega \in B_P \cup B_S$ and $\forall \varepsilon > 0$, define $E_\varepsilon(\omega) \equiv \begin{cases} \left|E(\omega)\right| & \left|E(\omega)\right| \geq \dfrac{\varepsilon}{2} \\ \dfrac{(E(\omega))^2}{\varepsilon} + \dfrac{\varepsilon}{4} & \left|E(\omega)\right| < \dfrac{\varepsilon}{2} \end{cases}$

and $J_\varepsilon(\mathbf{x}_n,\mathbf{x}_d) \equiv \displaystyle\int_{B_P \cup B_S} W(\omega) E_\varepsilon(\omega) d\omega$

Filter Designs

- Solutions for Solving Nonsmooth Optimization Problems
  - Now we approximate the problem as the following smooth optimization problem:

$$\min_{(\mathbf{x}_n,\mathbf{x}_d)} \quad J_\varepsilon(\mathbf{x}_n,\mathbf{x}_d) \equiv \int_{B_P \cup B_S} W(\omega) E_\varepsilon(\omega) d\omega$$

subject to $\hat{g}_1(\mathbf{x}_n,\mathbf{x}_d) = 0$

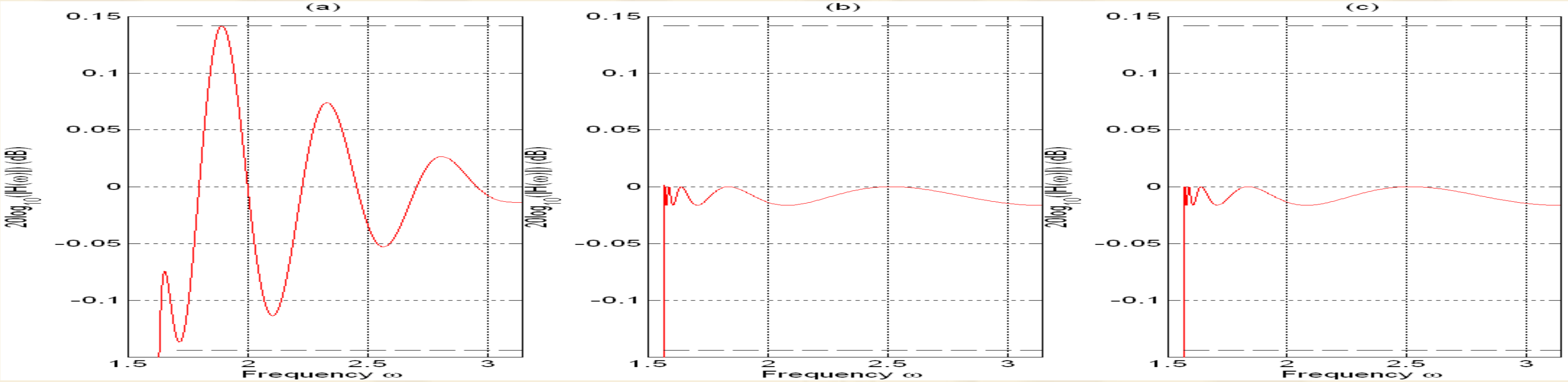
$$\hat{g}_2(\mathbf{x}_n,\mathbf{x}_d) = 0$$

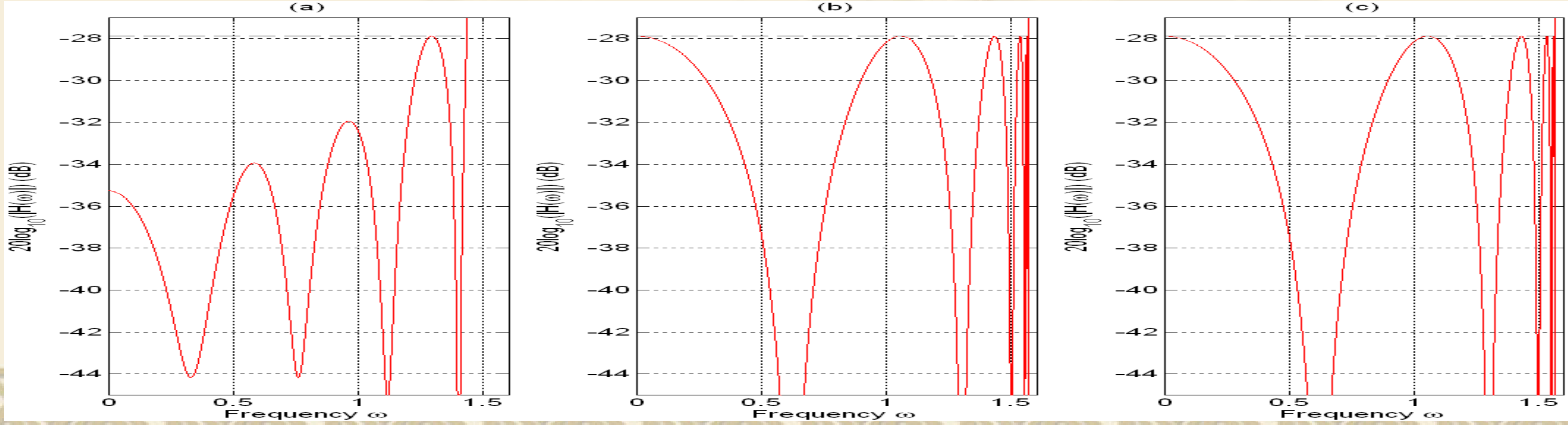
$$\hat{g}_3(\mathbf{x}_d) = 0$$

  - The optimization problem becomes a smooth optimization method and conventional Newton's method and gradient decent method can be applied for solving the problem.

Filter Designs

❖ Computer Numerical Simulation Results





Filter Designs

- FIR Linear Phase Quadrature mirror Filter (QMF) Design Problems
  - The highpass analysis filter: $H_1(z) = H_0(-z)$
  - The lowpass synthesis filter: $F_0(z) = 2H_0(z)$
  - The highpass synthesis filter: $F_1(z) = -2H_0(-z)$
  - Polyphase representation of the prototype filter:

    $H_0(z) \equiv E_0(z^2) + z^{-1}E_1(z^2)$
  - No aliasing distortion and phase distortion
  - Amplitude distortion: $T(z) = 4z^{-1}E_0(z^2)E_1(z^2) = 4z^{-(N-1)}E_0(z^2)E_0(z^{-2})$

Filter Designs

- FIR Linear Phase Quadrature mirror Filter (QMF) Design Problems
  - Define $\mathbf{x} \equiv \left[\delta_a, \quad \delta_p, \quad \delta_s, \quad h(0), \quad h(2), \quad \cdots, \quad h(N-2)\right]^T$

$$\boldsymbol{\eta}(\omega) \equiv \left[0, \quad 0, \quad 0, \quad 1, \quad e^{-j\omega}, \quad \cdots, \quad e^{-j\left(\frac{N}{2}-1\right)\omega} \right]^T$$

$$\mathbf{Q}(\omega) \equiv 8(\boldsymbol{\eta}(2\omega))^* (\boldsymbol{\eta}(2\omega))^T$$

  - Then $T(\omega) = 4e^{-j\omega(N-1)}\mathbf{x}^T (\boldsymbol{\eta}(2\omega))^* (\boldsymbol{\eta}(2\omega))^T \mathbf{x}$

and $|T(\omega)| = \left| \frac{1}{2}\mathbf{x}^T \mathbf{Q}(\omega)\mathbf{x} - 1 \right|$

Filter Designs

- FIR Linear Phase Quadrature mirror Filter (QMF) Design Problems
  - Define $\boldsymbol{\iota}_a \equiv \begin{bmatrix} 1, & 0, & \cdots, & 0 \end{bmatrix}^T$
  - Then the constraint on the aliasing distortion is

$$\frac{1}{2}\mathbf{x}^T\mathbf{Q}(\omega)\mathbf{x} - \boldsymbol{\iota}_a^T\mathbf{x} - 1 \le 0 \quad \forall\omega \in [-\pi, \pi]$$

$$-\frac{1}{2}\mathbf{x}^T\mathbf{Q}(\omega)\mathbf{x} - \boldsymbol{\iota}_a^T\mathbf{x} + 1 \le 0 \quad \forall\omega \in [-\pi, \pi]$$

Filter Designs

- FIR Linear Phase Quadrature mirror Filter (QMF) Design Problems
  - Define

$$\boldsymbol{\kappa}(\omega) \equiv 2\left[0, \quad 0, \quad 0, \quad \cos\left(\left(\frac{N-1}{2}\right)\omega\right), \quad \cos\left(\left(\frac{N-5}{2}\right)\omega\right), \quad \cdots, \quad \cos\left(\left(\frac{3-N}{2}\right)\omega\right)\right]^T$$

  - Then

$$H_0(\omega) = (\boldsymbol{\eta}(2\omega))^T \mathbf{x} + e^{-j\omega(N-1)}(\boldsymbol{\eta}(2\omega))^+ \mathbf{x}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}\left(\left[0, \quad 0, \quad 0, \quad e^{j\left(\frac{N-1}{2}\right)\omega}, \quad e^{j\left(\frac{N-5}{2}\right)\omega}, \quad \cdots, \quad e^{-j\left(\frac{N-3}{2}\right)\omega}\right]\mathbf{x} + \left[0, \quad 0, \quad 0, \quad e^{-j\left(\frac{N-1}{2}\right)\omega}, \quad e^{-j\left(\frac{N-5}{2}\right)\omega}, \quad \cdots, \quad e^{j\left(\frac{N-3}{2}\right)\omega}\right]\mathbf{x}\right)$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)}(\boldsymbol{\kappa}(\omega))^T \mathbf{x}$$

Filter Designs

- FIR Linear Phase Quadrature mirror Filter (QMF) Design Problems
  - Define $\boldsymbol{\iota}_p \equiv [0, \quad 1, \quad 0, \quad \cdots, \quad 0]^T$
  - Then the constraint on the maximum passband ripple magnitude of the prototype filter is

    $$\left|(\boldsymbol{\kappa}(\omega))^T \mathbf{x} - D(\omega)\right| \leq \boldsymbol{\iota}_p^T \mathbf{x} \ \forall \omega \in B_p$$
  - Define $\mathbf{A}_p(\omega) \equiv [\boldsymbol{\kappa}(\omega) - \boldsymbol{\iota}_p, \quad -\boldsymbol{\kappa}(\omega) - \boldsymbol{\iota}_p]^T$

    $$\mathbf{c}_p(\omega) \equiv [D(\omega), \quad -D(\omega)]^T$$
  - Then $\mathbf{A}_p(\omega)\mathbf{x} - \mathbf{c}_p(\omega) \leq \mathbf{0} \ \forall \omega \in B_p$

Filter Designs

- FIR Linear Phase Quadrature mirror Filter (QMF) Design Problems
  - Define $\boldsymbol{\iota}_s \equiv [0, \quad 0, \quad 1, \quad 0, \quad \cdots, \quad 0]^T$

$$\mathbf{A}_s(\omega) \equiv [\boldsymbol{\kappa}(\omega) - \boldsymbol{\iota}_s, \quad -\boldsymbol{\kappa}(\omega) - \boldsymbol{\iota}_s]^T$$

$$\mathbf{c}_s(\omega) \equiv [D(\omega), \quad -D(\omega)]^T$$

  - Similarly, the constraint on the maximum stopband ripple magnitude of the prototype filter is

$$\mathbf{A}_s(\omega)\mathbf{x} - \mathbf{c}_s(\omega) \leq \mathbf{0} \quad \forall \omega \in B_s$$

Filter Designs

- FIR Linear Phase Quadrature mirror Filter (QMF) Design Problems
  - Define $\mathbf{A}_b \equiv [\mathbf{I}, \quad \mathbf{0}]$
    $\mathbf{c}_b \equiv [\varepsilon_a, \quad \varepsilon_p, \quad \varepsilon_s]^T$
  - Then the specifications on the acceptable bounds on the maximum amplitude distortion of the filter bank, the maximum passband ripple magnitude and the maximum stopband ripple magnitude of the prototype filter is $\mathbf{A}_b\mathbf{x} - \mathbf{c}_b \leq \mathbf{0}$
  - The QMF design problem becomes:

$$\min_{\mathbf{x}} \quad f(\mathbf{x}) \equiv (\alpha \boldsymbol{\iota}_a + \beta \boldsymbol{\iota}_p + \gamma \boldsymbol{\iota}_s)^T \mathbf{x}$$

subject to

$$g_1(\mathbf{x},\omega) \equiv \frac{1}{2}\mathbf{x}^T\mathbf{Q}(\omega)\mathbf{x} - \boldsymbol{\iota}_a^T\mathbf{x} - 1 \leq 0 \quad \forall \omega \in [-\pi, \pi]$$

$$g_2(\mathbf{x},\omega) \equiv -\frac{1}{2}\mathbf{x}^T\mathbf{Q}(\omega)\mathbf{x} - \boldsymbol{\iota}_a^T\mathbf{x} + 1 \leq 0 \quad \forall \omega \in [-\pi, \pi]$$

$$g_3(\mathbf{x},\omega) \equiv \mathbf{A}_p(\omega)\mathbf{x} - \mathbf{c}_p(\omega) \leq \mathbf{0} \quad \forall \omega \in B_p$$

$$g_4(\mathbf{x},\omega) \equiv \mathbf{A}_s(\omega)\mathbf{x} - \mathbf{c}_s(\omega) \leq \mathbf{0} \quad \forall \omega \in B_s$$

$$g_5(\mathbf{x}) \equiv \mathbf{A}_b\mathbf{x} - \mathbf{c}_b \leq \mathbf{0}$$

Filter Designs

- Challenges of Nonconvex Optimization Problems
  - The feasible set is nonconvex.
  - There are many local minima. By using conventional gradient decent approaches, the optimization algorithms are usually stuck at these local minima and it is difficult to obtain the global minima of the optimization problems.

Filter Designs

- Filled function method for Solving Nonconvex Optimization Problems
  - Step 1: Initialize a minimum improvement factor ε, an accepted error ε', an initial search point $\tilde{\mathbf{x}}_1$, a positive definite matrix **R**, and an iteration index k=1.
  - Step 2: Find a local minimum of the following optimization Problem (**P$_\mathbf{f}$**) using conventional gradient decent approach with the initial search point $\tilde{\mathbf{x}}_k$.

$$\min_{\mathbf{x}} f(\mathbf{x}) \equiv \left(\alpha \boldsymbol{\iota}_a + \beta \boldsymbol{\iota}_p + \gamma \boldsymbol{\iota}_s\right)^T \mathbf{x}$$

$$g_1(\mathbf{x},\omega) \equiv \frac{1}{2}\mathbf{x}^T \mathbf{Q}(\omega)\mathbf{x} - \boldsymbol{\iota}_a^T \mathbf{x} - 1 \le 0 \quad \forall \omega \in [-\pi, \pi]$$

$$g_2(\mathbf{x},\omega) \equiv -\frac{1}{2}\mathbf{x}^T \mathbf{Q}(\omega)\mathbf{x} - \boldsymbol{\iota}_a^T \mathbf{x} + 1 \le 0 \quad \forall \omega \in [-\pi, \pi]$$

$$g_3(\mathbf{x},\omega) \equiv \mathbf{A}_p(\omega)\mathbf{x} - \mathbf{c}_p(\omega) \le \mathbf{0} \quad \forall \omega \in B_p$$

$$g_4(\mathbf{x},\omega) \equiv \mathbf{A}_s(\omega)\mathbf{x} - \mathbf{c}_s(\omega) \le \mathbf{0} \quad \forall \omega \in B_s$$

$$g_5(\mathbf{x}) \equiv \mathbf{A}_b \mathbf{x} - \mathbf{c}_b \le \mathbf{0}$$

$$g_6(\mathbf{x}) \equiv \boldsymbol{\iota}_a^T \left(\mathbf{x} - (1-\varepsilon)\tilde{\mathbf{x}}_k\right) \le 0$$

$$g_7(\mathbf{x}) \equiv \boldsymbol{\iota}_p^T \left(\mathbf{x} - (1-\varepsilon)\tilde{\mathbf{x}}_k\right) \le 0$$

$$g_8(\mathbf{x}) \equiv \boldsymbol{\iota}_s^T \left(\mathbf{x} - (1-\varepsilon)\tilde{\mathbf{x}}_k\right) \le 0$$

Filter Designs

- Filled function method for Solving Nonconvex Optimization Problems
  - Step 3: Find a local minimum of the following optimization Problem (**P$_H$**) using conventional gradient decent approach with the initial search point $\mathbf{x}_k^*$.

$$\min_{\mathbf{x}} \; H(\mathbf{x}) \equiv \left(\alpha\,\boldsymbol{\iota}_a + \beta\,\boldsymbol{\iota}_p + \gamma\,\boldsymbol{\iota}_s\right)^T \mathbf{x} + \frac{1}{\left(\mathbf{x}-\mathbf{x}_k^*\right)^T \mathbf{R}\left(\mathbf{x}-\mathbf{x}_k^*\right)}$$

$$g_1(\mathbf{x},\omega) \equiv \frac{1}{2}\mathbf{x}^T\mathbf{Q}(\omega)\mathbf{x} - \boldsymbol{\iota}_a^T\mathbf{x} - 1 \le 0 \quad \forall\omega\in[-\pi,\pi]$$

$$g_2(\mathbf{x},\omega) \equiv -\frac{1}{2}\mathbf{x}^T\mathbf{Q}(\omega)\mathbf{x} - \boldsymbol{\iota}_a^T\mathbf{x} + 1 \le 0 \quad \forall\omega\in[-\pi,\pi]$$

$$g_3(\mathbf{x},\omega) \equiv \mathbf{A}_p(\omega)\mathbf{x} - \mathbf{c}_p(\omega) \le \mathbf{0} \quad \forall\omega\in B_p$$

$$g_4(\mathbf{x},\omega) \equiv \mathbf{A}_s(\omega)\mathbf{x} - \mathbf{c}_s(\omega) \le \mathbf{0} \quad \forall\omega\in B_s$$

$$g_5(\mathbf{x}) \equiv \mathbf{A}_b\mathbf{x} - \mathbf{c}_b \le \mathbf{0}$$

$$g_6'(\mathbf{x}) \equiv \boldsymbol{\iota}_a^T\left(\mathbf{x} - (1-\varepsilon)\mathbf{x}_k^*\right) \le 0$$

$$g_7'(\mathbf{x}) \equiv \boldsymbol{\iota}_p^T\left(\mathbf{x} - (1-\varepsilon)\mathbf{x}_k^*\right) \le 0$$

$$g_8'(\mathbf{x}) \equiv \boldsymbol{\iota}_s^T\left(\mathbf{x} - (1-\varepsilon)\mathbf{x}_k^*\right) \le 0$$

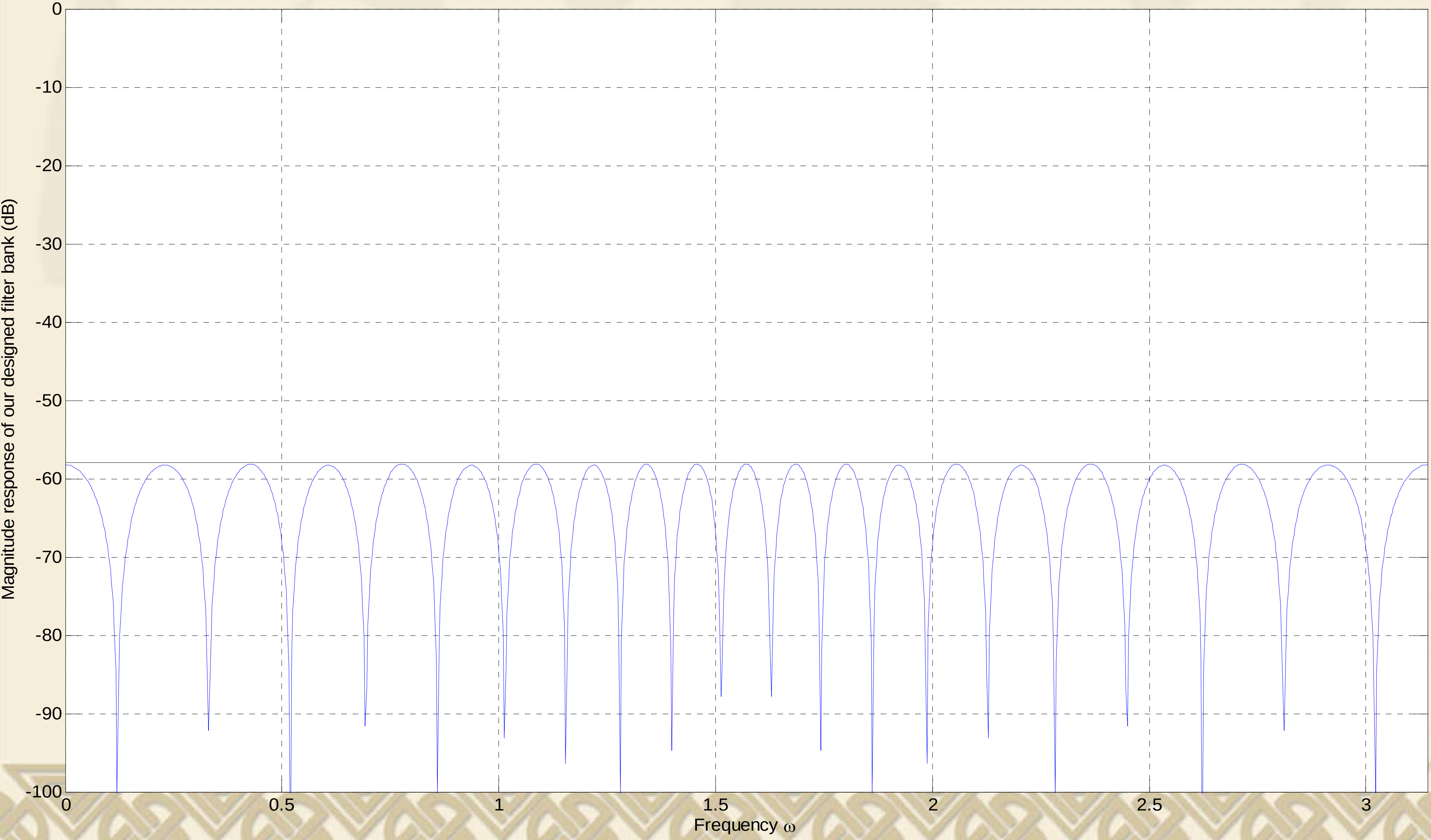
Filter Designs

- Filled function method for Solving Nonconvex Optimization Problems
  - Step 4: Iterate Step 2 and Step 3 until

$$\left\| \left(\alpha\,\boldsymbol{\iota}_a + \beta\,\boldsymbol{\iota}_p + \gamma\,\boldsymbol{\iota}_s\right)^T \left(\mathbf{x}_k^* - \mathbf{x}_{k-1}^*\right) \right\| \le \varepsilon'$$

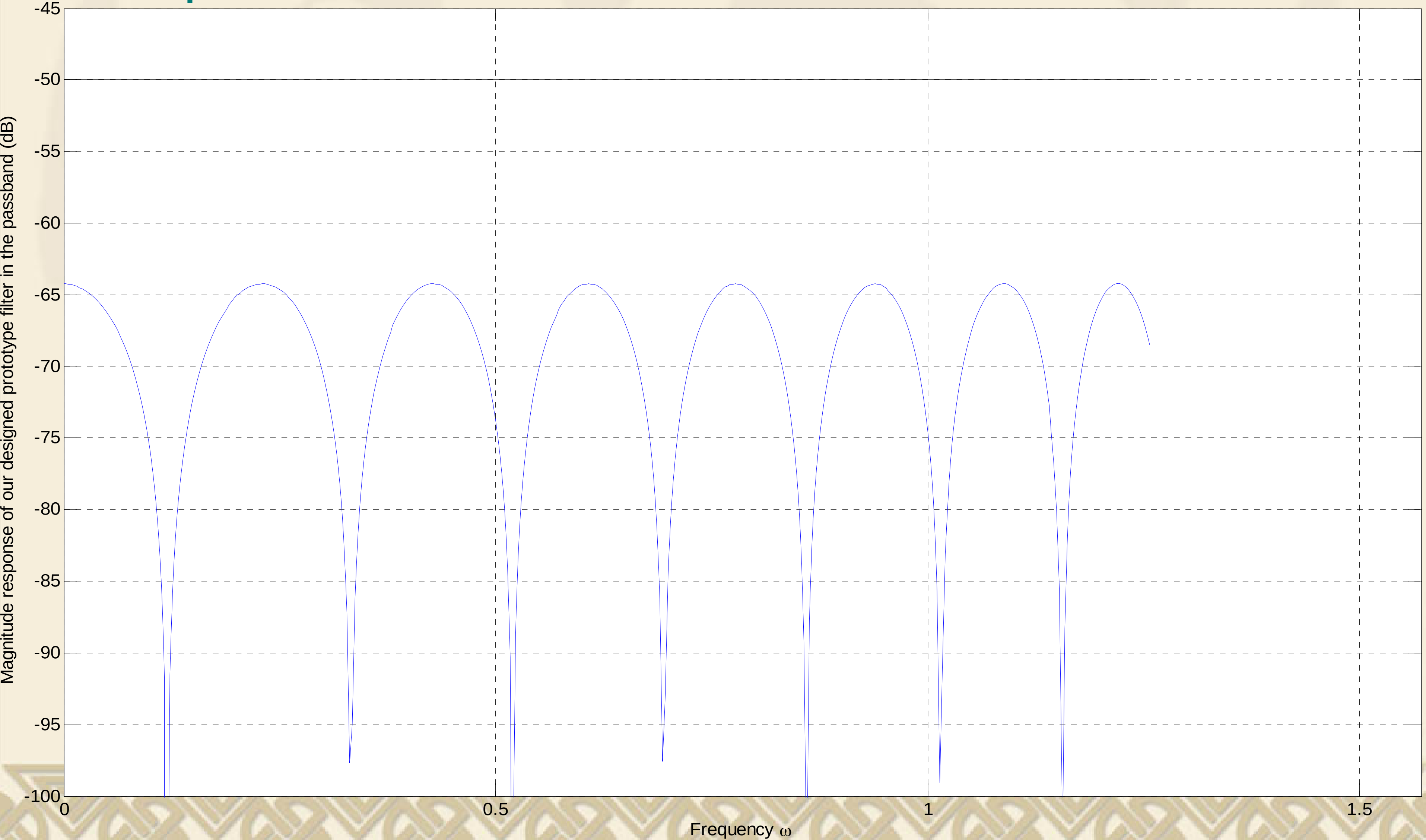
Filter Designs

- Computer Numerical Simulation Results



Filter Designs

❖ Computer Numerical Simulation Results



Filter Designs

- Computer Numerical Simulation Results

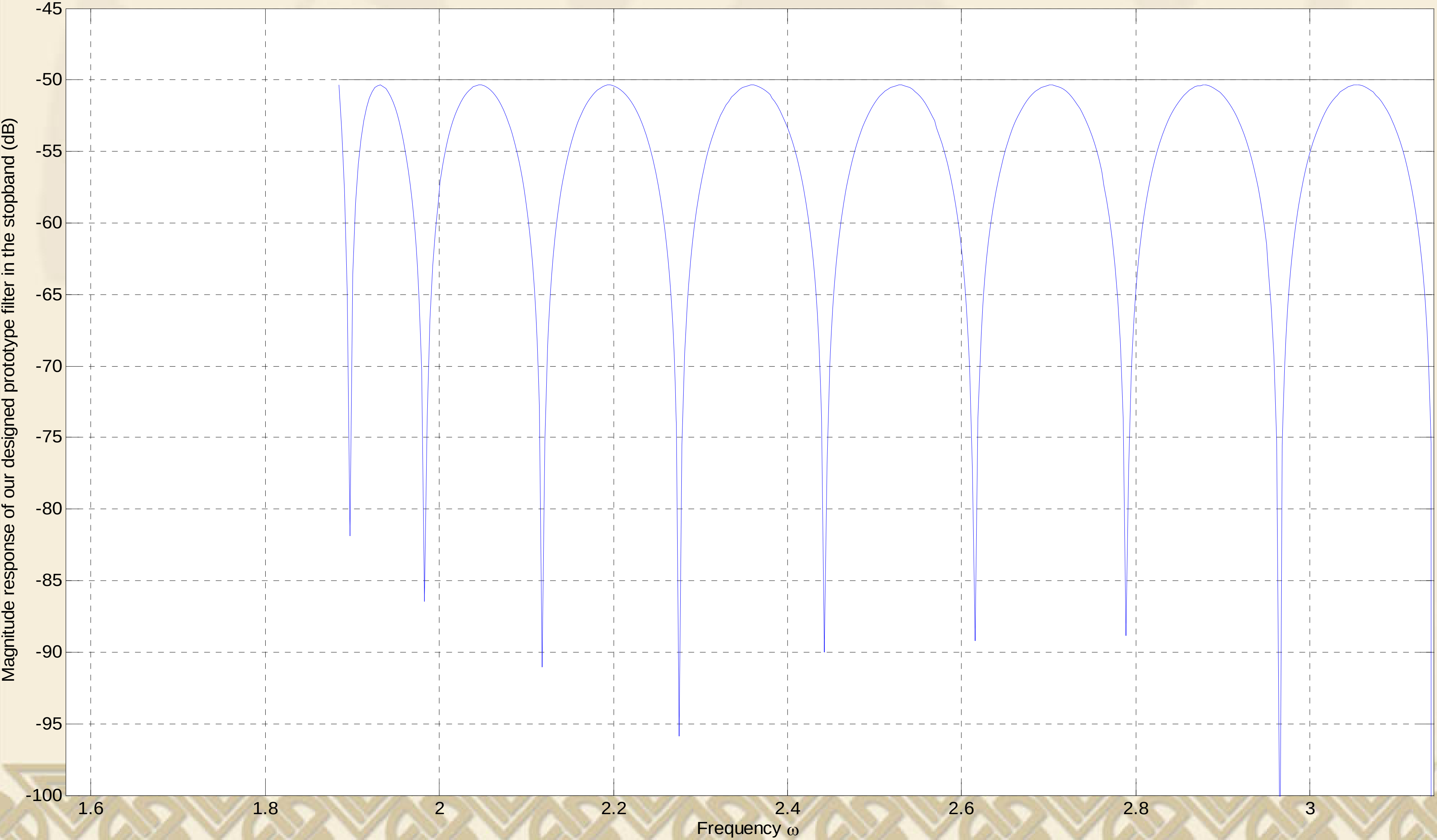


Image Resizing

- Two Dimensional Discrete Cosine Transform

$$F(u,v) \equiv \sqrt{\frac{2}{M}}\sqrt{\frac{2}{N}}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}\Lambda(u)\Lambda(v)\cos\left(\frac{\pi u(2n+1)}{2N}\right)\cos\left(\frac{\pi v(2m+1)}{2M}\right)f(n,m)$$

for $u = 0,\cdots,N'-1$ and $v = 0,\cdots,M'-1$

where

$$\Lambda(\xi) \equiv \begin{cases} \dfrac{1}{\sqrt{2}} & \xi = 0 \\ 1 & otherwise \end{cases}$$

Define

$$C(u,n) \equiv \sqrt{\frac{2}{N}}\Lambda(u)\cos\left(\frac{\pi u(2n+1)}{2N}\right)$$

$$\mathbf{S}(v,m) \equiv \sqrt{\frac{2}{M}}\Lambda(v)\cos\left(\frac{\pi v(2m+1)}{2M}\right)$$

Note that $\mathbf{f} \in \Re^{N\times M}$, $\mathbf{F} \in \Re^{N'\times M'}$, $\mathbf{C} \in \Re^{N'\times N}$ and $\mathbf{S} \in \Re^{M'\times M}$

Image Resizing

- Two Dimensional Discrete Cosine Transform
  - Note that $\mathbf{F} = \mathbf{C}\mathbf{f}\mathbf{S}^T$
  - If $N' \geq N$ and $M' \geq M$, then $rank(\mathbf{C}) = N$, $rank(\mathbf{S}) = M$ and $\mathbf{f} = (\mathbf{C}^T\mathbf{C})^{-1}\mathbf{C}^T\mathbf{F}\mathbf{S}(\mathbf{S}^T\mathbf{S})^{-1}$
  - If $N = N'$ and $M = M'$, then $\mathbf{C}$ and $\mathbf{S}$ are unitary.

Image Resizing

- Frames
  - A set of linear dependent vectors $\{\mathbf{e}_k\}$ that span a space V.
  - There exists two real numbers A>0 and B>0 such that

$$A\|\mathbf{v}\|^2 \le \sum_{\forall k} \langle \mathbf{v}, \mathbf{e}_k \rangle^2 \le B\|\mathbf{v}\|^2 \quad \forall \mathbf{v} \in V$$

  - Sets of the rows of $\mathbf{C}$ and $\mathbf{S}$ are bilinear frames.

Image Resizing

- DCT2 Based Image Enlargement Algorithm
  - Step 1: Divide the image into blocks with the size of each block being square
  - Step 2: $\mathbf{F} = \mathbf{C}\mathbf{f}\mathbf{S}^T$
  - Step 3: Compute IDCT2 of $\mathbf{F}$

Image Resizing

- Computer Simulation Results

Image Resizing

- Bilinear Tight Frame Design
  - Design $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{S}}$ such that $\tilde{\mathbf{f}} = \tilde{\mathbf{C}}\mathbf{f}\tilde{\mathbf{S}}^T$, $\tilde{\mathbf{C}}^T\tilde{\mathbf{C}} \approx \mathbf{I}_N$ and $\tilde{\mathbf{S}}^T\tilde{\mathbf{S}} \approx \mathbf{I}_M$
  - Problem formulation

$$\min_{(\tilde{\mathbf{C}},\tilde{\mathbf{S}})} \left\| \tilde{\mathbf{C}}^T\tilde{\mathbf{C}} - \mathbf{I}_N \right\| + \left\| \tilde{\mathbf{S}}^T\tilde{\mathbf{S}} - \mathbf{I}_M \right\|$$

Conclusions

- Many signal processing problems can be formulated as optimization problems.
- These optimization problems are indeed functional inequality constrained optimization problems, nonsmooth optimization problems and nonconvex optimization problems, which are challenge.
- Solving these optimization problems could improve performances of the corresponding signal processing systems. Hence, it is important to the signal processing community.

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- [3] Charlotte Yuk-Fan Ho, Bingo Wing-Kuen Ling, Yan-Qun Liu, Peter Kwong-Shun Tam and Kok-Lay Teo, “Optimum Design of Discrete-time Differentiators via Semi-infinite Programming Approach,” *IEEE Transactions on Instrumentation and Measurement*, vol. 57, no. 10, pp.2226-2230, 2008.

Questions and Answers

Thank you!

Let me think…